AN INVERSE PROBLEM OF THE FLUX FOR MINIMAL SURFACES

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Dedicated to Professor Masaru Takeuchi on his sixtieth birthday

Abstract. For a complete minimal surface in the Euclidean 3-space, the so-called flux vector corresponds to each end. The flux vectors are balanced, i.e., the sum of those over all ends are zero. Consider the following inverse problem: For each balanced n vectors, find an n-end catenoid which attains given vectors as flux. Here, an n-end catenoid is a complete minimal surface of genus 0 with ends asymptotic to the catenoids. In this paper, the problem is reduced to solving algebraic equation. Using this reduction, it is shown that, when $n = 4$, the inverse problem for 4-end catenoid has solutions for almost all balanced 4 vectors. Further obstructions for n-end catenoids with parallel flux vectors are also discussed.

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1. Introduction

An *n-end catenoid* is a complete immersed minimal surface of finite total curvature which has zero genus and n catenoid ends. It is considered as a conformal immersion $x\colon \hat{\mathbf{C}} \setminus \{q_1, \dots, q_n\} \to \mathbf{R}^3$, where $\hat{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$ and $q_1, \dots, q_n \in \hat{\mathbf{C}}$. Jorge-Meeks surfaces [JM] are typical ones. Recently, new examples of n-end catenoids have been found by [Kar], [L2], [Xu], [Ross1], [Ross2], [Kat] and [UY]. They contain examples with dihedral or Platonic symmetry groups. We also remark that for special classes of minimal surfaces with catenoid or flat ends, some systematic approach has been known (see [Pen], [Xi], [L1]).

1991 *Mathematics Subject Classification.* Primary 53A10; Secondary 53C42.

This research was supported in part by Grant-in-Aid for Scientific Research, the Ministry of Education, Sports and Culture, Japan.

In each end q_j $(j = 1, \dots, n)$ of an n-end catenoid, the *flux vector* is defined by

$$\varphi_j := \int_{\gamma_j} \vec{n}\, ds,$$

where γ_j is a curve surrounding q_j from the left and $\vec{n}$ the conormal such that $(\gamma_j', \vec{n})$ is positively oriented. Each flux vector φ_j is proportional to the limit normal vector $\nu(q_j)$ with respect to the end q_j and the scalar $w(q_j) := \varphi_j/4\pi\nu(q_j)$ is called the *weight* of the end q_j. It is well known that the flux vectors satisfy a "balancing" condition so called the *flux formula*

$$\sum_{j=1}^{n} \varphi_j = \sum_{j=1}^{n} 4\pi\, w(q_j)\, \nu(q_j) = 0.$$

It should be remarked that $w(q_j)$ may take a negative value.

Therefore, we consider an inverse problem of the flux formula as follows:

Problem. For given unit vectors $\{v_1, \dots, v_n\}$ in $\mathbf{R}^3$, and nonzero real numbers $\{a_1, \dots, a_n\}$ satisfying $\sum_{j=1}^{n} a_j v_j = 0$, is there an n-end catenoid $x\colon \hat{\mathbf{C}} \setminus \{q_1, \dots, q_n\} \to \mathbf{R}^3$ such that $\nu(q_j) = v_j$ and a_j is the weight at the end q_j?

We remark that Kusner also proposed a similar question (see [Ross1]). By the classification of Barbanel [Ba] and Lopez [L2], we can see that the answer for $n \le 3$ is "Yes" except for the case when two of $\{v_j\}_{j=1}^{n}$ coincide. For $n \ge 4$, the first author [Kat] gave an explicit formula for existence of an n-end catenoid with prescribed flux when $\hat{\mathbf{C}} \setminus \{q_1, \dots, q_n\}$ is conformally equivalent to the image of its Gauss map $\mathbf{S}^2 \setminus \{v_1, \dots, v_n\}$.

In this paper, we will generalize the formula in [Kat], and show some existence and non-existence results on the problem. In Section 2, we get the following:

Theorem A (Theorem 2.4). *For any pair $(\boldsymbol{v}, \boldsymbol{a})$ of unit vectors $\boldsymbol{v} = \{v_1, \dots, v_n\}$ in $\mathbf{R}^3$ and nonzero real numbers $\boldsymbol{a} = \{a_1, \dots, a_n\}$ satisfying $\sum_{j=1}^{n} a_j v_j = 0$, there is an evenly branched n-end catenoid $x\colon \hat{\mathbf{C}} \setminus \{q_1, \dots, q_n\} \to \mathbf{R}^3$ $(q_j \neq \infty)$ such that the induced metric is complete at the end q_j, $\nu(q_j) = v_j$ and a_j is the weight at the end q_j $(j = 1, \dots, n)$, if and only if there exist complex numbers $b_1, \dots, b_n$ satisfying the following conditions*:

$$\begin{cases} b_j \displaystyle\sum_{\substack{k=1\\k\neq j}}^{n} b_k \frac{p_k - p_j}{q_k - q_j} = a_j \\ b_j \displaystyle\sum_{\substack{k=1\\k\neq j}}^{n} b_k \frac{\bar{p}_j p_k + 1}{q_k - q_j} = 0 \end{cases} \qquad (j = 1, \dots, n),$$

where $p_j := \sigma(v_j)$, $\sigma : \mathbf{S}^2 \to \hat{\mathbf{C}}$ is the stereographic projection, and we assume $p_j \neq \infty$.

Moreover, the immersion x has no branch points if and only if the resultant $\Psi(P(z), Q(z))$ of the polynomials $P(z)$ and $Q(z)$ (defined by (2.13) and (2.12)) does not vanish.

We note here that the flux formula holds even if the surface allows branch points (see Remark 2.9).

In the case when an n-end catenoid has the same symmetry as its flux data, the construction is reduced to a routine work by virtue of our theorem, and one can construct all of the known examples (cf. [Kar], [L2], [Xu], [Ross1], [Ross2], [Kat], [UY], etc.) and far more new examples (cf. [KUY2]).

We also remark here that an n-end catenoid does not always have the symmetry of its flux data. In fact, there exists a flux data $(\boldsymbol{v}, \boldsymbol{a})$ such that any corresponding surface does not have the same symmetry as $(\boldsymbol{v}, \boldsymbol{a})$ (see Example 3.7(iii)).

On the other hand, for a certain flux data, there are no n-end catenoids realizing it. Indeed, there are no n-end catenoids with the flux data $(\boldsymbol{v}, \boldsymbol{a})$ satisfying one of the following conditions:

(1) $v_1 = v_2 = v_3 = \cdots = v_n$;
(2) $-v_1 = -v_2 = v_3 = \cdots = v_n$;
(3) $-v_1 = v_2 = \cdots = v_n$ and $\sum_{j=2}^{n-1} \sum_{k=j+1}^{n} a_j a_k \neq 0$;
(4) $(n = 4)$ $-v_1 = v_2$ and $v_3 = v_4 \neq \pm v_1$.

The first condition is well-known, and the third condition follows from the genus zero case of the second compatibility condition in [Per]. The fourth condition is new. These four obstructions are easily obtained as a corollary of Theorem A.

It is interesting to observe that, when the equality $\sum_{j=2}^{n-1} \sum_{k=j+1}^{n} a_j a_k = 0$ holds in (3) above, n-end catenoids which allow the deformation described by Lopez-Ros [LR] can be constructed (see Examples 4.7,4.8 and 4.9).

In spite of the above non-existence results, it seems that generic flux data are free of additional obstructions. We demonstrate it for $n = 4$, and get the following:

Theorem B (Theorems 3.3 and 3.6). *For almost all pair $(\boldsymbol{v}, \boldsymbol{a})$ of unit vectors $\boldsymbol{v} = \{v_1, v_2, v_3, v_4\}$ in $\mathbf{R}^3$ and nonzero real numbers $\boldsymbol{a} = \{a_1, a_2, a_3, a_4\}$ satisfying $\sum_{j=1}^{4} a_j v_j = 0$, there is a (non-branched) 4-end catenoid $x \colon \hat{\mathbf{C}} \setminus \{q_1, q_2, q_3, q_4\} \to \mathbf{R}^3$ such that $\nu(q_j) = v_j$ and a_j is the weight at the end q_j $(j = 1, 2, 3, 4)$, where ν is the Gauss map of x. Moreover, the number of such x is at most 4 up to rigid motions in $\mathbf{R}^3$. In particular, there exist 4-end catenoids with no symmetric properties.*

We remark here that $n = 4$ is the smallest number such that n-end catenoids have various conformal structures and that there exist mutually non-congruent n-end catenoids with the same flux data. Indeed, the upper estimate in the theorem above is sharp (see Example 3.7 and Figure 3.2).

To prove the first part of Theorem B, we give an explicit algorithm to construct 4-end catenoids with the prescribed flux by reducing it to solve a

certain algebraic equation of degree 4. However, to treat the case when $n \geq 5$, we shall have to do more complicated analysis (cf. [KUY1]).

The authors are very grateful to Dr. Wayne Rossman for valuable discussions and encouragements. They also thank to Prof. Osamu Kobayashi and Dr. Shin Nayatani for useful comments.

2. Reduction of the problem

For a positive integer n, we fix a Riemann surface

$$M^2 = \hat{\mathbf{C}} \setminus \{q_1, \dots, q_n\},$$

where $\hat{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$ and $q_1, \dots, q_n$ are mutually distinct points. A pair (g, ω) of a meromorphic function g and non-vanishing meromorphic 1-form ω on $\hat{\mathbf{C}}$ is called *Weierstrass data*. By the Enneper-Weierstrass representation formula, the map defined by

$$x := \operatorname{Re}\left(\int_{z_0}^{z} (1-g^2)\omega, \int_{z_0}^{z} i(1+g^2)\omega, \int_{z_0}^{z} 2g\omega\right) \tag{2.1}$$

is a branched minimal immersion on the universal covering on M^2, where z_0 is a fixed point on M^2. Any isolated degenerate point of the induced metric

$$ds^2 = (1+|g|^2)^2|\omega|^2 \tag{2.2}$$

is corresponding to a branch point of the map. The branched minimal immersion x is single-valued on M^2 if and only if

$$\begin{cases} \operatorname*{Res}_{z=q_j}(g\omega) \in \mathbf{R}, \\ \operatorname*{Res}_{z=q_j}(\omega) = -\overline{\operatorname*{Res}_{z=q_j}(g^2\omega)}, \end{cases} \qquad (j = 1, \dots, n), \tag{2.3}$$

where $\operatorname{Res}_{z=q_j}$ is the residue at $z = q_j$. Moreover, if ds^2 is a complete Riemannian metric on M^2, then the map x is a complete minimal immersion on M^2 with finite total curvature. Conversely, any complete conformal minimal immersion $x\colon M^2 \to \mathbf{R}^3$ with finite total curvature is constructed in such manner from the following Weierstrass data:

$$g = \frac{\partial x_3}{\partial x_1 - i\partial x_2}, \tag{2.4}$$

$$\omega = \partial x_1 - i\partial x_2, \tag{2.5}$$

where the function g is the stereographic projection of the Gauss map.

In this section, we rewrite the condition (2.3) into purely algebraic ones. We remark that the second fundamental form of the minimal immersion x is expressed by $-\omega \cdot dg - \overline{\omega \cdot dg}$ and its $(2,0)$-part $\omega \cdot dg$ is called the *Hopf differential* of the immersion x. The end q_j is called a *catenoid end* if the end is asymptotic to a catenoid by a suitable homothety, that is, the Gauss map has no branch point at the each end q_j and the Hopf differential $\omega \cdot dg$ has a pole of order -2 at q_j. The minimal immersion x is called an *n-end catenoid*

if all ends $q_1, \dots, q_n$ are catenoid ends. We also use a terminology *branched n-end catenoid* when the induced metric allows at most finite degenerate points. In particular, we call a branched n-end catenoid is *evenly branched* if all of its branch points are of even order. We regard non-branched n-end catenoids as special cases of evenly branched n-end catenoids.

First we prepare the following lemma:

Lemma 2.1. *Let $x\colon M^2 \to \mathbf{R}^3$ be a branched n-end catenoid. Then the degree of its Gauss map is at most $n-1$ and by a suitable motion in $\mathbf{R}^3$, the Weierstrass data given by* (2.4) *and* (2.5) *are taken to be satisfying the following conditions*:

(i) *ω has poles of order -2 on $\{q_1, \dots, q_n\}$.*

(ii) *g has no poles and branch points on $\{q_1, \dots, q_n\}$.*

Moreover, x has no branch points if and only if the degree of the Gauss map is $n-1$.

Proof. By a suitable motion in $\mathbf{R}^3$, we may assume that g has no poles on $\{q_1, \dots, q_n\}$. We apply the relation

$$\sum_{z \in Z(\omega)} \mathrm{Ord}_z(\omega) + \sum_{z \in S(\omega)} \mathrm{Ord}_z(\omega) = -\chi(\hat{\mathbf{C}}) = -2, \tag{2.6}$$

where $Z(\omega)$ and $S(\omega)$ are the set of zeros and the set of poles respectively. The assumption of catenoid ends yields that the Hopf differential $\omega \cdot dg$ has a pole of order -2 and dg has no zero at each end q_j. So ω has exactly order -2 at each end q_j. Therefore we have that

$$\sum_{z \in S(\omega)} \mathrm{Ord}_z(\omega) = -2n. \tag{2.7}$$

By (2.6) and (2.7), we have

$$\sum_{z \in Z(\omega)} \mathrm{Ord}_z(\omega) = 2n - 2. \tag{2.8}$$

On the other hand, since g has no pole at each end, any pole of g must be a zero of ω by (2.2). In particular, the inequality

$$\mathrm{Ord}_z(\omega) \geq -2 \, \mathrm{Ord}_z(g) \qquad (z \in Z(\omega)). \tag{2.9}$$

holds. Since the degree $\deg(g)$ of Gauss map is given by

$$\deg(g) = -\sum_{z \in S(g)} \mathrm{Ord}_z(g),$$

we have

$$\sum_{z \in Z(\omega)} \mathrm{Ord}_z(\omega) \geq 2 \deg(g). \tag{2.10}$$

By (2.8) and (2.10), we get

$$\deg(g) \leq n - 1. \tag{2.11}$$

Here, x has no branch points if and only if (2.9) is an equality, and hence the equality of (2.11) holds if and only if x has no branch points. $\square$

Lemma 2.2. *Let $M^2 = \hat{\mathbf{C}} \setminus \{q_1, \ldots, q_n\}$. Let g be a meromorphic function and ω a meromorphic* 1*-form on $\hat{\mathbf{C}}$ satisfying the conditions* (i) *and* (ii) *of the Lemma* 2.1*. Set $p_j := g(q_j)$. Assume $q_j \neq \infty$ and $p_j \neq \infty$ $(j = 1, \ldots, n)$. Then the symmetric tensor*

$$ds^2 = (1 + |g|^2)^2 |\omega|^2 \tag{2.2}$$

is a complete Riemannian metric on M^2 if and only if there exist two polynomials

$$Q(z) = \sum_{j=1}^{n} b_j \prod_{\substack{k=1 \\ k \neq j}}^{n} (z - q_k), \tag{2.12}$$

$$P(z) = \sum_{j=1}^{n} p_j b_j \prod_{\substack{k=1 \\ k \neq j}}^{n} (z - q_k) \tag{2.13}$$

$(b_1, \ldots, b_n \in \mathbf{C})$ *satisfying the following properties*:

(i) $\mathrm{Max}\{\deg(P), \deg(Q)\} = n - 1$.
(ii) *$P(z)$ and $Q(z)$ are irreducible.*
(iii) *$g(z) = P(z)/Q(z)$ and $\omega(z) = -\{\sum_{j=1}^{n} b_j/(z - q_j)\}^2 dz$.*

Proof. We suppose that ds^2 is a complete Riemannian metric on M^2. Since ds^2 is positive definite on M^2, we have from (2.2) that the inequality (2.9) turns to be an equality

$$\mathrm{Ord}_z(\omega) = -2\,\mathrm{Ord}_z(g) \qquad (z \in Z(\omega)\,). \tag{2.14}$$

By the same argument in the proof of the previous lemma, we have

$$\deg(g) = n - 1. \tag{2.15}$$

Since ω has only poles of order -2, poles and zeros of ω are all even order. Thus $\sqrt{\omega}$ is defined as a meromorphic section of the half canonical line bundle and has poles of order -1 on $P(\omega)$. Since ω has no pole at infinity, there exist complex numbers $b_1, \ldots, b_n \in \mathbf{C}$ such that

$$\sqrt{\omega} = i \left(\sum_{j=1}^{n} \frac{b_j}{z - q_j} \right) \sqrt{dz}.$$

Now we set

$$R(z) := \prod_{j=1}^{n} (z - q_j), \tag{2.16}$$

$$R_j(z) := \prod_{\substack{k=1 \\ k \neq j}}^{n} (z - q_k). \tag{2.17}$$

Then we have $\omega = -\left(\sum_{j=1}^{n} b_j R_j(z)\right)^2 / R(z)^2$. Hence, by (2.14), g can be written as

$$g = \frac{P(z)}{\sum_{j=1}^{n} b_j R_j(z)},$$

where $P(z)$ is a polynomial of order $n-1$. Clearly, $Q(z)$ defined by (2.12) satisfies $Q(z) = \sum_{j=1}^{n} b_j R_j(z)$. Moreover, we have (2.13) since $g(q_j) = p_j$. By (2.15), $P(z)$ and $Q(z)$ are irreducible and $\mathrm{Max}\{\deg(P), \deg(Q)\} = n-1$. On the other hand, the symmetric tensor ds^2 induced from such two polynomials $P(z)$ and $Q(z)$ by (2.2) is obviously a complete Riemannian metric on M^2. $\square$

The following proposition reduces the conditions (2.12) and (2.13) to a purely algebraic condition, which plays essential roles in this paper:

Proposition 2.3. *Let $n \geq 2$ be an integer and $q_1, \dots, q_n, p_1, \dots, p_n, b_1, \dots, b_n$ complex numbers. Set $g(z) := P(z)/Q(z)$ and $\omega(z) := -\{\sum_{j=1}^{n} b_j/(z-q_j)\}^2 dz$, where $P(z)$ and $Q(z)$ are polynomials defined by* (2.12) *and* (2.13) *respectively. Then the branched minimal immersion*

$$x = \mathrm{Re}\left(\int_{z_0}^{z} (1-g^2)\omega, \int_{z_0}^{z} i(1+g^2)\omega, \int_{z_0}^{z} 2g\omega\right)$$

is single-valued on the Riemann surface $M^2 = \hat{\mathbf{C}} \setminus \{q_1, \dots, q_n\}$ if and only if the following conditions hold:

$$\text{(2.18)} \qquad \begin{cases} b_j \displaystyle\sum_{\substack{k=1 \\ k \neq j}}^{n} b_k \frac{p_j - p_k}{q_j - q_k} \in \mathbf{R} \\ b_j \displaystyle\sum_{\substack{k=1 \\ k \neq j}}^{n} b_k \frac{\bar{p}_j p_k + 1}{q_j - q_k} = 0 \end{cases} \qquad (j = 1, \dots, n).$$

Proof. By (2.12) and (2.13), one can easily get the following identities

$$\operatorname*{Res}_{z=q_j}(\omega) = -2b_j \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{b_k}{q_j - q_k},$$

$$\operatorname*{Res}_{z=q_j}(g\omega) = -b_j \sum_{\substack{k=1 \\ k \neq j}}^{n} b_k \frac{p_j + p_k}{q_j - q_k},$$

$$\operatorname*{Res}_{z=q_j}(g^2\omega) = -2b_j p_j \sum_{\substack{k=1 \\ k \neq j}}^{n} b_k \frac{p_k}{q_j - q_k}.$$

Thus the conditions (2.3) can be rewritten as

$$
(2.3')\quad \begin{cases} b_j \displaystyle\sum_{\substack{k=1\\k\neq j}}^{n} b_k \frac{p_j+p_k}{q_j-q_k} \in \mathbf{R} \\ b_j \displaystyle\sum_{\substack{k=1\\k\neq j}}^{n} \frac{b_k}{q_j-q_k} = -\overline{b_j p_j \displaystyle\sum_{\substack{k=1\\k\neq j}} b_k \frac{p_k}{q_j-q_k}} \end{cases} \qquad (j=1,\dots,n).
$$

If we set

$$
(2.19)\quad \begin{cases} A_j := b_j \displaystyle\sum_{\substack{k=1\\k\neq j}}^{n} \frac{b_k}{q_j-q_k} \\ B_j := b_j \displaystyle\sum_{\substack{k=1\\k\neq j}}^{n} b_k \frac{p_k}{q_j-q_k} \end{cases} \qquad (j=1,\dots,n),
$$

then (2.3) is equivalent to the following condition:

$$
(2.20)\quad \begin{cases} p_j A_j + B_j \in \mathbf{R} \\ A_j + \bar{p}_j \overline{B}_j = 0 \end{cases} \qquad (j=1,\dots,n).
$$

It can be easily seen that (2.20) implies that $p_j A_j$ and B_j are both real numbers. Hence (2.20) reduces to the following condition:

$$
(2.21)\quad \begin{cases} p_j A_j - B_j \in \mathbf{R} \\ A_j + \bar{p}_j B_j = 0 \end{cases} \qquad (j=1,\dots,n),
$$

which is equivalent to the desired condition (2.18). $\square$

Next we consider the flux formula on n-end catenoid. We fix a Riemann surface

$$
M^2 = \hat{\mathbf{C}} \setminus \{q_1,\dots,q_n\},
$$

where $q_1,\dots,q_n$ are mutually distinct points. Let $x\colon M^2 \to \mathbf{R}^3$ be a branched n-end catenoid. Then the *flux vector* at an end q_j is defined by

$$
(2.22)\qquad \varphi_j := \int_{\gamma_j} \vec{n}\, ds,
$$

where γ_j is a circle surrounding q_j from the left, and $\vec{n}$ the conormal such that $(\gamma_j', \vec{n})$ is positively oriented. The flux vector is independent of choice of a circle γ_j. Each flux vector φ_j is proportional to the limit normal vector $\nu(q_j)$ with respect to the end q_j, and the real number $a_j := \varphi_j/4\pi\nu(q_j)$ is called the *weight* of the end q_j. One can easily verify that the Hopf differential $\omega \cdot dg$ has the following Laurent expansion at each end q_j:

$$
(2.23)\qquad \omega \cdot dg = \left\{ \frac{a_j}{(z-q_j)^2} + \cdots \right\} dz^2,
$$

where a_j is the weight at the end q_j. By Lemma 2.1, we may assume that g has no poles and ω has poles of order -2 on $\{q_1,\dots,q_n\}$. Then, by Lemma 2.2,

there exist complex numbers $p_1, \dots, p_n$ and $b_1, \dots, b_n$ such that $g(z) = P(z)/Q(z)$ and $\omega(z) = -\{\sum_{j=1}^n b_j/(z-q_j)\}^2 dz$, where $P(z)$ and $Q(z)$ are polynomials defined by (2.12) and (2.13) respectively. Then, by (2.23), we have the identities

$$a_j = b_j \sum_{\substack{k=1\\k\neq j}}^{n} b_k \frac{p_j - p_k}{q_j - q_k} \qquad (j = 1, \dots, n). \tag{2.24}$$

We remark that the reality of a_j follows from the conditions (2.18) and (2.24). Since the limit normal vector $\nu(q_j)$ with respect to the end q_j is expressed by

$$\nu(q_j) = \left(\frac{2\,\mathrm{Re}(p_j)}{|p_j|^2+1}, \frac{2\,\mathrm{Im}(p_j)}{|p_j|^2+1}, \frac{|p_j|^2-1}{|p_j|^2+1}\right),$$

as the inverse stereographic image of p_j, the flux formula stated in the introduction is rewritten as

$$\sum_{j=1}^{n} a_j \frac{|p_j|^2-1}{|p_j|^2+1} = 0, \qquad \sum_{j=1}^{n} a_j \frac{\bar{p}_j}{|p_j|^2+1} = 0. \tag{2.25}$$

As an application of Lemmas 2.1, 2.2 and Proposition 2.3, we have the following reduction theorem for the inverse problem of the flux formula:

Theorem 2.4. *Let $n \geq 2$ be an integer, $q_1, \dots, q_n, p_1, \dots, p_n, b_1, \dots, b_n$ complex numbers, and $a_1, \dots, a_n$ real numbers. Set $g(z) := P(z)/Q(z)$ and $\omega(z) := -\{\sum_{j=1}^n b_j/(z-q_j)\}^2 dz$, where $P(z)$ and $Q(z)$ are polynomials defined by* (2.12) *and* (2.13) *respectively. Then the map*

$$x := \mathrm{Re}\left(\int_{z_0}^{z} (1-g^2)\omega, \int_{z_0}^{z} i(1+g^2)\omega, \int_{z_0}^{z} 2g\omega\right)$$

is an evenly branched n-end catenoid defined on $M^2 = \hat{\mathbf{C}} \setminus \{q_1, \dots, q_n\}$ and has the flux vector

$$\varphi_j = 4\pi a_j \cdot \left(\frac{2\,\mathrm{Re}(p_j)}{|p_j|^2+1}, \frac{2\,\mathrm{Im}(p_j)}{|p_j|^2+1}, \frac{|p_j|^2-1}{|p_j|^2+1}\right)$$

at each end q_j if and only if the following condition holds:

$$\begin{cases} b_j \displaystyle\sum_{\substack{k=1\\k\neq j}}^{n} b_k \frac{p_k - p_j}{q_k - q_j} = a_j \\ b_j \displaystyle\sum_{\substack{k=1\\k\neq j}}^{n} b_k \frac{\bar{p}_j p_k + 1}{q_k - q_j} = 0 \end{cases} \qquad (j = 1, \dots, n). \tag{2.26}$$

Moreover, suppose $\mathrm{Max}\{\deg(P), \deg(Q)\} = n-1$, *and $P(z)$ and $Q(z)$ are irreducible. Then x has no branch points and is an n-end catenoid. Conversely, any evenly branched n-end catenoid is constructed in such manner.*

Proof. We set real numbers A_j and B_j by (2.19). Then, (2.26) is rewritten as

$$\begin{cases} \quad p_jA_j - B_j = a_j \\ -(A_j + \bar{p}_jB_j) = 0 \end{cases} \qquad (j = 1, \ldots, n).$$

Hence the first assertion follows immediately from (2.21) and (2.24). Conversely, we fix an evenly branched n-end catenoid with Weierstrass data (g, ω). Then the order of ω is even everywhere. Thus $\sqrt{\omega}$ is defined as a meromorphic section of the half canonical bundle. By the same argument of the proof of Lemma 2.2, we have the expression $g(z) := P(z)/Q(z)$ and $\omega(z) := -\{\sum_{j=1}^{n} b_j/(z-q_j)\}^2 dz$, where $P(z)$ and $Q(z)$ are polynomials defined by (2.12) and (2.13) respectively. This proves the second assertion. □

Remark 2.5. The Weierstrass data of any evenly branched n-end catenoid have the form as in Proposition 2.3. Therefore, it is branched if and only if the resultant of $P(z)$ and $Q(z)$ does not vanish. This algebraic equation is expected to have zeros of codimension 1. Indeed it is true in the case $n = 4$ as we will see in Section 3.

Remark 2.6. When $q_j = rp_j$ $(j = 1, \ldots, n)$, Theorem 2.4 reduces to the results in the first author [Kat]. In this case, the system (2.26) reduces to

$$\begin{cases} \dfrac{1}{r} b_j \displaystyle\sum_{\substack{k=1\\k\neq j}}^{n} b_k = a_j \\ \dfrac{1}{r} b_j \displaystyle\sum_{\substack{k=1\\k\neq j}}^{n} b_k \frac{\overline{p_j}p_k + 1}{p_k - p_j} = 0 \end{cases} \qquad (j = 1, \ldots, n).$$

Moreover, the surface has no branch points if and only if $\sum_{j=1}^{n} b^j \neq 0$. Many known and new examples of n-end catenoids can be constructed from this formula, and also from our formula (2.26) (cf. [KUY2]).

Remark 2.7. When $p_n \neq p_j$ $(j = 1, \ldots, n-1)$, we may assume $p_n = q_n = \infty$ without loss of generality. Under this assumption, since $p_j \neq \infty$ $(j = 1, \ldots, n-1)$ holds automatically, it is easy to see that the equation (2.26) can

be rewritten as the following version:

$$(2.27)\qquad \begin{cases} b_j\left(\displaystyle\sum_{\substack{k=1\\k\neq j}}^{n-1} b_k\frac{p_k-p_j}{q_k-q_j}+b_n\right)=a_j & (j=1,\dots,n-1),\\ b_n\displaystyle\sum_{k=1}^{n-1} b_k=a_n, \\ b_j\left(\displaystyle\sum_{\substack{k=1\\k\neq j}}^{n-1} b_k\frac{\bar p_j p_k+1}{q_k-q_j}+\bar p_j b_n\right)=0 & (j=1,\dots,n-1),\\ -b_n\displaystyle\sum_{k=1}^{n-1} p_k b_k=0. \end{cases}$$

In this situation, the polynomials $P(z)$, $Q(z)$ and $R(z)$ are replaced naturally as follows:

$$P(z)=\sum_{j=1}^{n-1} p_j b_j\prod_{\substack{k=1\\k\neq j}}^{n-1}(z-q_k)-b_n\prod_{k=1}^{n-1}(z-q_k),$$

$$Q(z)=\sum_{j=1}^{n-1} b_j\prod_{\substack{k=1\\k\neq j}}^{n-1}(z-q_k),$$

$$R(z)=\prod_{k=1}^{n-1}(z-q_k).$$

By easy calculation, we get the following

Corollary 2.8. *The assertion of Theorem* 2.4 *holds even if we replace the condition* (2.26) *by the following condition:*

$$(2.28)\qquad \begin{cases} b_j\displaystyle\sum_{\substack{k=1\\k\neq j}}^{n} b_k\frac{1}{q_j-q_k}=a_j\frac{\bar p_j}{|p_j|^2+1}\\ b_j\displaystyle\sum_{\substack{k=1\\k\neq j}}^{n} b_k\frac{p_j+p_k}{q_j-q_k}=a_j\frac{|p_j|^2-1}{|p_j|^2+1} \end{cases}\qquad (j=1,\dots,n).$$

Remark 2.9. Summing up the equations (2.28) for $j=1,\dots,n$, we get the flux formula (2.25) for any evenly branched n-end catenoids. However, the flux formula itself is still valid for minimal syrfaces with branch points of odd order. In fact, by straightforward calculation, the flux vector defined by (2.22) is written as

$$\varphi_j=-\operatorname{Im}\left(\int_{\gamma_j}(1-g^2)\omega,\int_{\gamma_j}i(1+g^2)\omega,\int_{\gamma_j}2g\omega\right).$$

The flux formula is obvious from the point of view.

3. 4-END CATENOIDS OF GENERIC TYPE

Let $x\colon \hat{\mathbf{C}} \setminus \{q_1, \dots, q_n\} \to \mathbf{R}^3$ be an *n-end catenoid*, and set $v_j := \nu(q_j)$ ($j = 1, \dots, n$), where ν is the Gauss map. Then, as we saw in the previous sections, a family $\boldsymbol{v} = \{v_1, \dots, v_n\}$ of n unit vectors in $\mathbf{R}^3$ must satisfy the following condition.

$$\text{(F.}n\text{)} \qquad \sum_{j=1}^{n} a_j v_j = 0 \qquad \text{for some nonzero real numbers} \quad a_1, \dots, a_n.$$

Now, we classify arrangements of $\boldsymbol{v} = \{v_1, \dots, v_n\}$ to the following three types:

TYPE I : $\boldsymbol{v} = \{v_1, \dots, v_n\}$ satisfies (F.n) and $\dim\langle v_1, \dots, v_n\rangle = 1$.
TYPE II : $\boldsymbol{v} = \{v_1, \dots, v_n\}$ satisfies (F.n) and $\dim\langle v_1, \dots, v_n\rangle = 2$.
TYPE III : $\boldsymbol{v} = \{v_1, \dots, v_n\}$ satisfies (F.n) and $\dim\langle v_1, \dots, v_n\rangle = 3$.

We call a (branched) n-end catenoid is of TYPE I (resp. II, III), if $\boldsymbol{v}$ is of TYPE I (resp. II, III).

The following facts are already known (e.g. [M], [Ba], [L2], [Kat]):

(1) *There are no* 1*-end catenoids.*
(2) *Any* 2*-end catenoid is the catenoid. Of course, it is of TYPE I.*
(3) *There are no* 3*-end catenoids of TYPE I. Consequently, any* 3*-end catenoid is of TYPE II. More precisely, for any unit vectors v_1, v_2, v_3 of TYPE II and nonzero real numbers a_1, a_2, a_3 satisfying $\sum_{j=1}^{3} a_j v_j = 0$, there exists an essentially unique* 3*-end catenoid $x\colon \hat{\mathbf{C}} \setminus \{q_1, q_2, q_3\} \to \mathbf{R}^3$ which satisfies $\nu(q_j) = v_j$ and the weight $w(q_j) = a_j$ ($j = 1, 2, 3$).*

From these results, the moduli of at most 3-end catenoids is understood completely.

In this section, we restrict our attention to 4-end catenoids of TYPE III that is a generic type. In particular, we give some upper estimates for the numbers $N_C(\boldsymbol{v}, \boldsymbol{a})$ of congruent classes of 4-end catenoids with given $(\boldsymbol{v}, \boldsymbol{a})$, and a method to construct these surfaces.

First we recall that if there is a 4-end catenoid $x\colon \hat{\mathbf{C}} \setminus \{q_1, q_2, q_3, q_4\} \to \mathbf{R}^3$ such that $\nu(q_j) = v_j$ and $w(q_j) = a_j$ ($j = 1, 2, 3, 4$), then, by (2.26), there are nonzero complex numbers b_1, b_2, b_3, b_4 satisfying

$$A \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

with

$$
(3.1)\qquad A := \begin{pmatrix} 0 & \dfrac{\bar{p}_1 p_2 + 1}{q_2 - q_1} & \dfrac{\bar{p}_1 p_3 + 1}{q_3 - q_1} & \dfrac{\bar{p}_1 p_4 + 1}{q_4 - q_1} \\ \dfrac{\bar{p}_2 p_1 + 1}{q_1 - q_2} & 0 & \dfrac{\bar{p}_2 p_3 + 1}{q_3 - q_2} & \dfrac{\bar{p}_2 p_4 + 1}{q_4 - q_2} \\ \dfrac{\bar{p}_3 p_1 + 1}{q_1 - q_3} & \dfrac{\bar{p}_3 p_2 + 1}{q_2 - q_3} & 0 & \dfrac{\bar{p}_3 p_4 + 1}{q_4 - q_3} \\ \dfrac{\bar{p}_4 p_1 + 1}{q_1 - q_4} & \dfrac{\bar{p}_4 p_2 + 1}{q_2 - q_4} & \dfrac{\bar{p}_4 p_3 + 1}{q_3 - q_4} & 0 \end{pmatrix},
$$

where $p_j := \sigma(v_j)$ $(j = 1, 2, 3, 4)$ and $\sigma \colon S^2 \to \hat{\mathbf{C}}$ is the stereographic projection from the north pole, that is, they satisfy the following identity:

$$
v_j = \left(\frac{2\,\mathrm{Re}(p_j)}{|p_j|^2 + 1}, \frac{2\,\mathrm{Im}(p_j)}{|p_j|^2 + 1}, \frac{|p_j|^2 - 1}{|p_j|^2 + 1} \right) \qquad (j = 1, \dots, n).
$$

Clearly, it holds that $\det A = 0$.

To get upper estimates for $N_C(\boldsymbol{v}, \boldsymbol{a})$, we discuss the rank of the matrix A. We remark that $\operatorname{rank} A$ is invariant under both the conformal actions of the domain and the rigid motions in $\mathbf{R}^3$.

When we consider v of TYPE III, we may assume $p_4 = q_4 = \infty$ without loss of generality, and we can use the formula (2.27) in place of (2.26). Under this assumption, the matrix A is given by

$$
(3.1')\qquad A = \begin{pmatrix} 0 & \dfrac{\bar{p}_1 p_2 + 1}{q_2 - q_1} & \dfrac{\bar{p}_1 p_3 + 1}{q_3 - q_1} & \bar{p}_1 \\ \dfrac{\bar{p}_2 p_1 + 1}{q_1 - q_2} & 0 & \dfrac{\bar{p}_2 p_3 + 1}{q_3 - q_2} & \bar{p}_2 \\ \dfrac{\bar{p}_3 p_1 + 1}{q_1 - q_3} & \dfrac{\bar{p}_3 p_2 + 1}{q_2 - q_3} & 0 & \bar{p}_3 \\ -p_1 & -p_2 & -p_3 & 0 \end{pmatrix}.
$$

Lemma 3.1. *Let $\boldsymbol{v} = \{v_1, v_2, v_3, v_4\}$ be unit vectors of TYPE III. Then the number $N_q(\boldsymbol{v})$ of solutions $\boldsymbol{q} = \{q_1, q_2, q_3, q_4\}$ of the equation $\det A = 0$ is at most 4 up to Möbius transformations.*

Proof. Set $p_j := \sigma(v_j)$ as before. We may assume $p_4 = q_4 = \infty$ without loss of generality, and it follows from direct computation that

$$
\det A = \frac{|p_1|^2 |\bar{p}_2 p_3 + 1|^2 q_1^4 + \cdots}{(q_1 - q_2)^2 (q_2 - q_3)^2 (q_3 - q_1)^2}.
$$

Since we assume $\boldsymbol{v}$ is of TYPE III, it is clear that $|p_1|^2 |\bar{p}_2 p_3 + 1|^2 \neq 0$. Therefore the number of q_1 satisfying $\det A = 0$ is at most 4 for any choice of q_2, q_3, namely $N_q(\boldsymbol{v}) \leq 4$. $\square$

Proposition 3.2. $\operatorname{rank} A = 3$ *if and only if $\boldsymbol{v}$ is of TYPE III.*

Proof. When $\boldsymbol{v}$ is of TYPE III, we can easily see that

$$\det \begin{pmatrix} 0 & \dfrac{\bar{p}_2 p_3 + 1}{q_3 - q_2} & \bar{p}_2 \\ \dfrac{\bar{p}_3 p_2 + 1}{q_2 - q_3} & 0 & \bar{p}_3 \\ -p_2 & -p_3 & 0 \end{pmatrix} = \frac{\bar{p}_3 p_2 - \bar{p}_2 p_3}{q_2 - q_3} \neq 0.$$

Hence we have $\operatorname{rank} A = 3$ for any q_1, q_2, q_3 satisfying $\det A = 0$.

On the other hand, when $\boldsymbol{v}$ is of TYPE I or II, by the remark above, we may assume p_1, p_2, p_3, p_4 are real numbers. In this case, since A is skew-symmetric, if $\det A = 0$ and $A \neq 0$, then we have $\operatorname{rank} A = 2$.

Now, our assertion has been proved. □

Theorem 3.3. *For any* $\boldsymbol{v} = \{v_1, v_2, v_3, v_4\}$ *of TYPE III and* $\boldsymbol{a} = \{a_1, a_2, a_3, a_4\}$ *satisfying* $\sum_{j=1}^{4} a_j v_j = 0$, $N_C(\boldsymbol{v}, \boldsymbol{a}) \leq 4$. *Namely, the number of* 4*-end catenoids with the same* $(\boldsymbol{v}, \boldsymbol{a})$ *of TYPE III is at most* 4.

Proof. From the proof of the proposition above, $\dim \operatorname{Ker} A = 1$ for any q_1, q_2, q_3 satisfying $\det A = 0$. Note that ${}^t(b_1, b_2, b_3, b_4) \in \operatorname{Ker} A - \{0\}$. Moreover, (b_1, b_2, b_3, b_4) satisfies $b_4(b_1 + b_2 + b_3) = a_4$ also. Hence, if $(b_1, b_2, b_3, b_4) = (\beta_1, \beta_2, \beta_3, \beta_4)$ is a solution , then the other solution is $(b_1, b_2, b_3, b_4) = -(\beta_1, \beta_2, \beta_3, \beta_4)$ and both of these solutions give the same Weierstrass data. Therefore, it is clear that, for any q_1, q_2, q_3 chosen above, the number of 4-end catenoids is at most 1. Now we get the estimate $N_C(\boldsymbol{v}, \boldsymbol{a}) \leq N_q(\boldsymbol{v}) \times 1 \leq 4$. □

Corollary 3.4. *Any* 4*-end catenoid of TYPE III is isolated in the sense of Rosenberg* [Rose].

Proof. Since $N_C(\boldsymbol{v}, \boldsymbol{a})$ is finite and any deformation moving flux is not an ϵ-C^1-variation, our assertion is clear. □

By solving the equation (2.27) with $n = 4$, $q_2 = p_2$ and $q_3 = p_3$ directly, we get the following method to construct 4-end catenoids of TYPE III.

For given $(\boldsymbol{v}, \boldsymbol{a})$, set $p_j := \sigma(v_j)$ $(j = 1, 2, 3, 4)$ as before, and set

$$A(t) := \begin{pmatrix} 0 & \dfrac{\bar{p}_1 p_2 + 1}{p_2 - t} & \dfrac{\bar{p}_1 p_3 + 1}{p_3 - t} & \bar{p}_1 \\ \dfrac{\bar{p}_2 p_1 + 1}{t - p_2} & 0 & \dfrac{\bar{p}_2 p_3 + 1}{p_3 - p_2} & \bar{p}_2 \\ \dfrac{\bar{p}_3 p_1 + 1}{t - p_3} & \dfrac{\bar{p}_3 p_2 + 1}{p_2 - p_3} & 0 & \bar{p}_3 \\ -p_1 & -p_2 & -p_3 & 0 \end{pmatrix}$$

(Remark that $A(q_1) = A|_{q_2=p_2, q_3=p_3}$),

$$\begin{aligned}
\Phi(t) &:= (p_2 - p_3)^2 (t - p_2)^2 (t - p_3)^2 \det A(t) \\
&= \det \begin{pmatrix} 0 & -(\bar{p}_1 p_2+1)(t-p_3) & -(\bar{p}_1 p_3+1)(t-p_2) & \bar{p}_1 (t-p_2)(t-p_3) \\ (\bar{p}_2 p_1+1)(p_2-p_3) & 0 & -(\bar{p}_2 p_3+1)(t-p_2) & \bar{p}_2 (p_2-p_3)(t-p_2) \\ (\bar{p}_3 p_1+1)(p_2-p_3) & (\bar{p}_3 p_2+1)(t-p_3) & 0 & \bar{p}_3 (p_2-p_3)(t-p_3) \\ -p_1 & -p_2 & -p_3 & 0 \end{pmatrix} \\
&= |p_1|^2 |\bar{p}_2 p_3 + 1|^2 t^4 + \cdots ,
\end{aligned}$$

and

$$B_1(t) := (\bar{p}_3 p_2 - \bar{p}_2 p_3)(p_2 - p_3)(t - p_2)(t - p_3).$$

If $\det A(t) = 0$ and $B_1(t) \neq 0$, then $\operatorname{Ker} A(t)$ is generated by ${}^t(B_1(t),\ B_2(t),\ B_3(t),\ B_4(t))$, where

$$\begin{pmatrix} B_2(t) \\ B_3(t) \\ B_4(t) \end{pmatrix} := A' \begin{pmatrix} (\bar{p}_2 p_1 + 1)(p_2 - p_3)(t - p_3) \\ (\bar{p}_3 p_1 + 1)(p_2 - p_3)(t - p_2) \\ -p_1 (t - p_2)(t - p_3) \end{pmatrix},$$

$$A' := \begin{pmatrix} -|p_3|^2 (p_2 - p_3) & \bar{p}_2 p_3 (p_2 - p_3) & \bar{p}_3 (\bar{p}_2 p_3 + 1)(p_2 - p_3) \\ \bar{p}_3 p_2 (p_2 - p_3) & -|p_2|^2 (p_2 - p_3) & -\bar{p}_2 (\bar{p}_3 p_2 + 1)(p_2 - p_3) \\ p_3 (\bar{p}_3 p_2 + 1) & -p_2 (\bar{p}_2 p_3 + 1) & -|\bar{p}_2 p_3 + 1|^2 \end{pmatrix}.$$

(The matrix A' is column-equivalent to the inverse of the 3×3 submatrix which results by deleting the first row and column of the matrix $A(t)$.)

Note that $B_1(q_1) \neq 0$ holds for any solution q_1 of the equation $\Phi(t) = 0$. Indeed, if $B_1(q_1) = 0$, then we have

$$(\bar{p}_3 p_2 - \bar{p}_2 p_3)(p_2 - p_3)(q_1 - p_2)(q_1 - p_3) = 0.$$

However, since we assume v is of TYPE III,

$$\begin{aligned}
\Phi(p_2) &= |p_3|^2 |\bar{p}_1 p_2 + 1|^2 (p_2 - p_3)^4 \neq 0, \\
\Phi(p_3) &= |p_2|^2 |\bar{p}_1 p_3 + 1|^2 (p_2 - p_3)^4 \neq 0,
\end{aligned}$$

namely $q_1 \neq p_2, p_3$. Hence the equality above does not happen. Assume $\prod_{k=2}^4 B_k(q_1) \neq 0$ and $\sum_{j=1}^3 B_j(q_1) \neq 0$. Then, by straightforward calculation, we see that the solutions of the equation (2.27) with $n = 4$, $q_2 = p_2$ and $q_3 = p_3$ are given by

$$\begin{cases} q_1 : \text{a solution of the equation} \quad \Phi(t) = 0, \\ q_2 := p_2, \quad q_3 := p_3, \quad q_4 := \infty, \\ b_1 := (\pm) B_1(q_1) \sqrt{\dfrac{a_4}{B_4(q_1) \sum_{j=1}^3 B_j(q_1)}}, \\ b_j := \dfrac{B_j(q_1)}{B_1(q_1)} b_1 \qquad (j = 2, 3, 4). \end{cases}$$

If q_1 satisfies $\prod_{k=2}^4 B_k(q_1) = 0$ or $\sum_{j=1}^3 B_j(q_1) = 0$, then there are no solution $(\boldsymbol{q}, \boldsymbol{b})$ of the equation (2.27) with such q_1, since we assume $a_j \neq 0$ $(j = 1, 2, 3, 4)$. Set

$$\begin{aligned}
\tilde{P}(z) &:= p_1 B_1(t)(z-p_2)(z-p_3) + p_2 B_2(t)(z-t)(z-p_3) \\
&\qquad + p_3 B_3(t)(z-t)(z-p_2) - B_4(t)(z-t)(z-p_2)(z-p_3), \\
\tilde{Q}(z) &:= B_1(t)(z-p_2)(z-p_3) + B_2(t)(z-t)(z-p_3) + B_3(t)(z-t)(z-p_2), \\
\tilde{R}(z) &:= (z-t)(z-p_2)(z-p_3).
\end{aligned}$$

Note here that

$$\begin{aligned}
\tilde{P}(z)|_{t=q_1} &= \frac{B_1(q_1)}{b_1} P(z)|_{q_2=p_2, q_3=p_3}, \\
\tilde{Q}(z)|_{t=q_1} &= \frac{B_1(q_1)}{b_1} Q(z)|_{q_2=p_2, q_3=p_3}, \\
\tilde{R}(z)|_{t=q_1} &= R(z)|_{q_2=p_2, q_3=p_3}.
\end{aligned}$$

Then it is easy to see that the $(\boldsymbol{q}, \boldsymbol{b})$ above gives the following Weierstrass data of a 4-end catenoid realizing $(\boldsymbol{v}, \boldsymbol{a})$:

$$g(z) = \left.\frac{\tilde{P}(z)}{\tilde{Q}(z)}\right|_{t=q_1}, \qquad \omega = -\frac{a_4}{B_4(q_1)\sum_{j=1}^3 B_j(q_1)} \left(\left.\frac{\tilde{Q}(z)}{\tilde{R}(z)}\right|_{t=q_1} \right)^2 dz. \tag{3.2}$$

Let $\Psi(t)$ be the resultant of $\tilde{P}(z)$ and $\tilde{Q}(z)$. The surface given by the data above has no branch points if and only if q_1 satisfies $\Psi(q_1) \neq 0$.

We can construct all of the 4-end catenoids of TYPE III by this algorithm.

Now, we observe a typical

Example 3.5. Let ζ_3 be a primitive root of the equation $z^3 = 1$. For special values $p_1 = 1/\sqrt{2}$, $p_2 = \zeta_3/\sqrt{2}$ and $p_3 = \zeta_3^2/\sqrt{2}$, by direct computation, we have

$$\begin{aligned}
\Phi(t) &= \frac{3}{8}\left(t - \frac{1}{\sqrt{2}}\right)^2 (t+\sqrt{2})^2, \\
B_1(t) &= \frac{3}{2\sqrt{2}}\left(t^2 + \frac{1}{\sqrt{2}}t + \frac{1}{2}\right), \\
B_2(t) &= -\frac{3\zeta_3^2}{4\sqrt{2}}\left(t^2 - \frac{4\zeta_3+1}{\sqrt{2}}t - \zeta_3\right), \\
B_3(t) &= -\frac{3\zeta_3}{4\sqrt{2}}\left(t^2 - \frac{4\zeta_3^2+1}{\sqrt{2}}t - \zeta_3^2\right), \\
B_4(t) &= \frac{3}{4\sqrt{2}}\left(t^2 + \frac{1}{\sqrt{2}}t + 2\right).
\end{aligned}$$

For one solution $1/\sqrt{2}$ of the equation $\Phi(t)=0$, we have

$$
\begin{aligned}
B_j\left(\frac{1}{\sqrt{2}}\right) &= \frac{9}{4\sqrt{2}} \qquad\qquad (j=1,2,3,4),\\
\sum_{j=1}^{3} B_j\left(\frac{1}{\sqrt{2}}\right) &= \frac{27}{4\sqrt{2}} \neq 0,\\
\tilde{P}(z)|_{t=1/\sqrt{2}} &= -\frac{9}{4\sqrt{2}}(z^3-\sqrt{2}),\\
\tilde{Q}(z)|_{t=1/\sqrt{2}} &= \frac{27}{4\sqrt{2}}z^2,\\
\tilde{R}(z)|_{t=1/\sqrt{2}} &= z^3-\frac{1}{2\sqrt{2}},
\end{aligned}
$$

from which it follows that $\Phi(1/\sqrt{2})\neq 0$. Now, (3.2) with these data gives an Enneper-Weierstrass representation of a tetrahedrally symmetric 4-end catenoid.

On the other hand, for the other solution $-\sqrt{2}$, we have

$$
\begin{aligned}
B_j(-\sqrt{2}) &= \frac{9\zeta_3^{j-1}}{4\sqrt{2}} \qquad\qquad (j=1,2,3,4),\\
\sum_{j=1}^{3} B_j(-\sqrt{2}) &= 0,\\
\tilde{P}(z)|_{t=-\sqrt{2}} &= -\frac{9}{4\sqrt{2}}\left(z^3+\frac{3}{\sqrt{2}}z^2+\frac{3}{2}z-\frac{1}{2\sqrt{2}}\right),\\
\tilde{Q}(z)|_{t=-\sqrt{2}} &= -\frac{27}{8}\left(z+\frac{1}{\sqrt{2}}\right),\\
\tilde{R}(z)|_{t=-\sqrt{2}} &= z^3+\frac{3}{\sqrt{2}}z^2+\frac{3}{2}z+\frac{1}{\sqrt{2}},
\end{aligned}
$$

from which it follows that $\Phi(-\sqrt{2})\neq 0$. Now, the data

$$
g(z)=\left.\frac{\tilde{P}(z)}{\tilde{Q}(z)}\right|_{t=q_1}, \qquad \omega=-\left(\left.\frac{\tilde{Q}(z)}{\tilde{R}(z)}\right|_{t=q_1}\right)^2 dz.
$$

gives an Enneper-Weierstrass representation of a complete minimal surface with 4 flat ends. It is easy to see that this surface is also tetrahedrally symmetric (cf. [Br]).

From this consideration, it is clear that the tetrahedrally symmetric 4-end catenoid is unique up to homothety.

Now we will prove our second main theorem in Section 1. Note here that $\Phi(t)$, $B_j(t)$ $(j=1,2,3,4)$ and $\Psi(t)$ are polynomials of t whose coefficients are also polynomials of p_1, p_2, p_3, $\bar{p}_1$, $\bar{p}_2$ and $\bar{p}_3$.

Theorem 3.6. *For almost all pair $(\boldsymbol{v}, \boldsymbol{a})$ of unit vectors $\boldsymbol{v} = \{v_1, v_2, v_3, v_4\}$ in $\mathbf{R}^3$ and nonzero real numbers $\boldsymbol{a} = \{a_1, a_2, a_3, a_4\}$ satisfying $\sum_{j=1}^4 a_j v_j = 0$, there is a 4-end catenoid $x\colon \hat{\mathbf{C}} \setminus \{q_1, q_2, q_3, q_4\} \to \mathbf{R}^3$ such that $\nu(q_j) = v_j$ and the weight $w(q_j) = a_j$ $(j = 1, 2, 3, 4)$.*

Proof. Set $p_j := \sigma(v_j)$ and assume $p_4 = q_4 = \infty$ as before. Then $\boldsymbol{v}$ is of TYPE III if and only if

$$\begin{aligned} D := {} & (\bar{p}_2 p_3 - p_2 \bar{p}_3)(\bar{p}_3 p_1 - p_3 \bar{p}_1)(\bar{p}_1 p_2 - p_1 \bar{p}_2) \times \{(|p_1|^2 - 1)(\bar{p}_2 p_3 - p_2 \bar{p}_3) \\ & \quad + (|p_2|^2 - 1)(\bar{p}_3 p_1 - p_3 \bar{p}_1) + (|p_3|^2 - 1)(\bar{p}_1 p_2 - p_1 \bar{p}_2)\} \\ & \neq 0. \end{aligned}$$

(D can be obtained from the multiplication of the 3×3 minor determinants of the 3×4 matrix (v_1, v_2, v_3, v_4), where $v_4 = {}^t(0, 0, 1)$). In this case, what we have only to do is to show the existence of a solution q_1 of the equation $\Phi(t) = 0$ satisfying $\prod_{k=2}^4 B_k(q_1) \neq 0$, $\sum_{j=1}^3 B_j(q_1) \neq 0$ and $\Psi(q_1) \neq 0$. Let $\Omega(t)$ be the remainder upon division of $(\Psi(t) \prod_{k=2}^4 B_k(t) \sum_{j=1}^3 B_j(t))^4$ by $\Phi(t)$. Clearly, $\Omega(t) \not\equiv 0$ if and only if there exists at least one solution q_1 of the equation $\Phi(t) = 0$ satisfying $\prod_{k=2}^4 B_k(q_1) \neq 0$, $\sum_{j=1}^3 B_j(q_1) \neq 0$ and $\Psi(q_1) \neq 0$.

Now, since there exists the tetrahedrally symmetric 4-end catenoid, it is clear that at least one coefficient Ω_0 of $\Omega(t)$ does not vanish for special values $p_1 = 1/\sqrt{2}$, $p_2 = \zeta_3/\sqrt{2}$ and $p_3 = \zeta_3^2/\sqrt{2}$ (see Example 3.5). Remark here that each coefficient of $\Omega(t)$ is a rational function of $p_1, p_2, p_3, \bar{p}_1, \bar{p}_2, \bar{p}_3$, i.e. it is real analytic. Hence, we see that $\Omega(t) \not\equiv 0$ for almost all p_1, p_2, p_3. Now we have proved that, for any $p_1, p_2, p_3, p_4 (= \infty)$ satisfying $(\Omega_0 D)(p_1, p_2, p_3, \bar{p}_1, \bar{p}_2, \bar{p}_3) \neq 0$ and $\boldsymbol{a} = \{a_1, a_2, a_3, a_4\}$ such that $\sum_{j=1}^4 a_j \sigma^{-1}(p_j) = 0$, there is at least one 4-end catenoid $x\colon \hat{\mathbf{C}} \setminus \{q_1, q_2, q_3, q_4\} \to \mathbf{R}^3$ such that $\nu(q_j) = \sigma^{-1}(p_j)$ and $w(q_j) = a_j$. $\square$

We know that we cannot remove "almost". Indeed, there are additional obstructions for the existence of 4-(or n-)end catenoids. We will consider these in the following section.

Before concluding this section, let us observe a significant example which includes a deformation from the Jorge-Meeks 4-noid to the tetrahedrally symmetric 4-end catenoid. Moreover, we will remark there that the symmetry of the flux data of an 4-end catenoid does not always imply the symmetry of the surface.

Example 3.7. We will now try to construct a 4-end catenoid with the following data;

$$\begin{cases} p_1 = p, \quad p_2 = -p, \quad p_3 = p^{-1} i, \quad p_4 = -p^{-1} i, \\ a_1 = a_2 = a_3 = a_4 = 1, \end{cases} \tag{3.3}$$

where p is a positive real number. The corresponding flux data $(\boldsymbol{v}, \boldsymbol{a})$ is invariant under the action of the group

$$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \subset O(3).$$

We may put

$$q_1 = q, \quad q_2 = -q, \quad q_3 = q^{-1}i, \quad q_4 = -q^{-1}i,$$

for a nonzero complex number q ($q^4 \neq -1$) without loss of generality, and the determinant of the matrix A in (3.1) is computed as

$$\det A = \left[\left\{ \frac{2q(q^2-1)}{q^4+1} \right\}^2 - \left(\frac{p^2-1}{2p} \right)^2 \right] \cdot \left[\left\{ \frac{2q(q^2+1)}{q^4+1} \right\}^2 + \left(\frac{p^2-1}{2p} \right)^2 \right].$$

Thus $\det A = 0$ if and only if one of the following four equations is satisfied:

$$(3.4_\pm) \qquad \frac{2q(q^2-1)}{q^4+1} = \pm \frac{p^2-1}{2p};$$

$$(3.5_\pm) \qquad \frac{2q(q^2+1)}{q^4+1} = \pm \frac{p^2-1}{2p} i.$$

Note here that $q = \alpha$ is a solution of (3.4_+) if and only if $q = -\alpha$ (resp. $q = \pm\alpha^{-1}i$) is a solution of (3.4_-) (resp. $(3.5_\pm)$), and that the corresponding solutions $(\boldsymbol{q}, \boldsymbol{b})$ of (2.26) give the Weierstrass data of the surfaces congruent to each other. Therefore we may only consider (3.4_+). It is easy to see that the equation (3.4_+) has a real solution q if and only if $c^{-1} \leq p \leq c$, where $c := (\sqrt{6} + \sqrt{2})/2$.

Let q be a solution of the equation (3.4_+). Now, since our data (3.3) is of TYPE III, rank$A$ must be 3 (except for the case $p = 1$). It is clear that the nonzero vector ${}^t(q, q, p, p) \in$Ker$A$, and hence it spans Ker$A$. Therefore we can set

$$b_1 = b_2 = rq, \qquad b_3 = b_4 = rp,$$

and, by (2.26), we get

$$r^2 = \frac{q^4+1}{q\{p(q^4+1) + 2q(p^2q^2+1)\}}.$$

Define a surface $x_q \colon \hat{\mathbf{C}} \setminus \{q, -q, q^{-1}i, -q^{-1}i\} \to \mathbf{R}^3$ by these data, whose Weierstrass data is given by

$$(3.6) \qquad g(z) := \frac{(pq^2 - q^{-1})z^2 + (p+q)}{z\{(p+q)z^2 - (pq^2 - q^{-1})\}},$$

$$(3.7) \qquad \omega := -r^2 \left[\frac{2z\{(p+q)z^2 - (pq^2 - q^{-1})\}}{(z^2 - q^2)(z^2 + q^{-2})} \right]^2 dz.$$

Then x_q is a branched conformal minimal immersion such that

$$\begin{cases} g(q) = p, \quad g(-q) = -p, \quad g(q^{-1}i) = p^{-1}i, \quad g(-q^{-1}i) = -p^{-1}i, \\ w(q) = w(-q) = w(q^{-1}i) = w(-q^{-1}i) = 1, \end{cases}$$

except for the case when $q = -c^{\pm 1}$ (i.e. $r = \infty$). It is easy to see that

(i) x_q is branched if and only if $q = -1$ ($p = 1$);

(ii) The normalized surface $\tilde{x}_q$ defined by the same g as x_q and $\tilde{\omega} := \omega/r^2$ has 4 flat ends (i.e. $a_j = 0$ for any j) if and only if $q = -c^{\pm 1}$ and $p = c^{\mp 1}$.

(iii) In the other cases, x_q is a non-branched 4-end catenoid. In particular, when $c^{-1} \leq p \leq c$, any solution q of (3.4_+) is real, and x_q has the same symmetry as its flux data (3.3). On the other hand, when $0 < p < c^{-1}$ or $c < p$, any solution q of (3.4_+) can take neither a real nor a purely imaginary value, and the isometry group of x_q is $\mathbf{Z}_2 \times \mathbf{Z}_2$ which is smaller than that of its flux data (3.3).

In particular,

(iv) x_1 is the Jorge-Meeks 4-noid;

(v) x_c is the tetrahedrally symmetric 4-end catenoid.

By (iii), (iv) and (v), we see that the family $\{x_q;\ 1 \leq q \leq c\}$ gives a deformation from the Jorge-Meeks 4-noid to the tetrahedrally symmetric one. In other words, the Jorge-Meeks 4-noid and the tetrahedrally symmetric one are included in the same connected component of the moduli of 4-end catenoids.

Furthermore, we can check that, for any real number p such that $1 < p < c$, there are four real numbers $q^{(1)}, q^{(2)}, q^{(3)}$ and $q^{(4)}$ satisfying

$$q^{(1)} > q^{(2)} > 1 > 0 > q^{(3)}(= -\frac{1}{q^{(1)}}) > q^{(4)}(= -\frac{1}{q^{(2)}}) > -1$$

and (3.4_+) with $q = q^{(\ell)}$ ($\ell = 1, 2, 3, 4$). Hence there are four 4-end catenoids $x_{q^{(1)}}$, $x_{q^{(2)}}$, $x_{q^{(3)}}$ and $x_{q^{(4)}}$ which have the same flux data. It can be easily observed that these four surfaces are not congruent to each other. This concludes that our estimate $N_C(\boldsymbol{v}, \boldsymbol{a}) \leq 4$ in Theorem 3.3 is sharp.

These situations are demonstrated in Figure 3.1. Figure 3.2 shows the image of x_q for various values of q with the same flux.

4. 4-end catenoids of special type and additional obstructions

In this section, we consider the cases of TYPEs I and II. First, we will prove that the same assertions as in Theorem 3.3 and Corollary 3.4 hold also in the case of TYPE II.

Let $N_C(\boldsymbol{v}, \boldsymbol{a})$, $N_q(\boldsymbol{v})$ and A be as in the previous section (see Lemma 3.1 etc.).

Lemma 4.1. *Let $\boldsymbol{v} = \{v_1, v_2, v_3, v_4\}$ be a family of unit vectors of TYPE II. Then $N_q(\boldsymbol{v}) \leq 2$.*

FIGURE 3.1. Example 3.7.

Proof. Set $p_j \colon = \sigma(v_j)$ $(j = 1, 2, 3, 4)$ as before, where σ is the stereographic projection. We may assume p_1, p_2, p_3, p_4 are real numbers without loss of generality. Set $p_{jk} := p_j p_k + 1$ for convenience. It follows from direct computation that

$$\det A = \left\{ \frac{p_{12}p_{34}}{(q_1 - q_2)(q_3 - q_4)} - \frac{p_{13}p_{24}}{(q_1 - q_3)(q_2 - q_4)} + \frac{p_{14}p_{23}}{(q_1 - q_4)(q_2 - q_3)} \right\}^2 .$$

We may also assume $q_1 = -q_2 \neq 0, \pm 1$ and $q_3 = -q_4 = 1$ without loss of generality, and we have

$$\det A = \frac{q_1 \Phi_{II}(q_1 + q_1^{-1})}{4(q_1 - 1)^2 (q_1 + 1)^2},$$

where

$$\Phi_{II}(t) = p_{12}p_{34}t^2 + 4(p_{13}p_{24} - p_{14}p_{23})t + 8(p_{13}p_{24} + p_{14}p_{23}) - 4p_{12}p_{34}.$$

Since we assume $\boldsymbol{v}$ is of TYPE II, it is clear that at least one of the coefficients of $\Phi_{II}(t)$ does not vanish. Therefore the number of $q_1 + q_1^{-1}$ satisfying $\det A = 0$ is at most 2. Since $\hat{\mathbf{C}} \setminus \{\pm q_1, \pm 1\}$ and $\hat{\mathbf{C}} \setminus \{\pm q_1^{-1}, \pm 1\}$ are conformal to each other by a Möbius transformation, we get $N_q(\boldsymbol{v}) \leq 2$. $\square$

Theorem 4.2. *For any $\boldsymbol{v} = \{v_1, v_2, v_3, v_4\}$ of TYPE II and $\boldsymbol{a} = \{a_1, a_2, a_3, a_4\}$ satisfying $\sum_{j=1}^4 a_j v_j = 0$, $N_C(\boldsymbol{v}, \boldsymbol{a}) \leq 4$. Namely, the number of 4-end catenoids with the same $(\boldsymbol{v}, \boldsymbol{a})$ of TYPE II is at most 4.*

Proof. From the proof of Proposition 3.2, $\dim \operatorname{Ker} A = 2$ for any q_1, q_2, q_3, q_4 satisfying $\det A = 0$. Note that ${}^t(b_1, b_2, b_3, b_4) \in \operatorname{Ker} A - \{0\}$. We may assume

(a) $p = 1.2$, $q \approx 1.0976$

(b) $p = 1.2$, $q \approx 10.815$

(c) $p = 1.2$, $q \approx -0.91078$

(d) $p = 1.2$, $q \approx -0.09246$

FIGURE 3.2. Non-congruent four surfaces with the same flux.

$p_{12}p_{34}(p_1 - p_2)(p_3 - p_4) \neq 0$ without loss of generality. By putting $q_1 = -q_2$ and $q_3 = -q_4 = 1$ into (2.26), we have

$$\begin{cases} b_3 = \dfrac{2}{p_{34}} \left(-\dfrac{p_{14}}{q_1 + 1} b_1 + \dfrac{p_{24}}{q_1 - 1} b_2 \right) \\ b_4 = -\dfrac{2}{p_{34}} \left(-\dfrac{p_{13}}{q_1 - 1} b_1 + \dfrac{p_{23}}{q_1 + 1} b_2 \right), \end{cases} \tag{4.1}$$

$$\begin{cases} b_3 \left(\dfrac{p_1 - p_3}{q_1 - 1} b_1 - \dfrac{p_2 - p_3}{q_1 + 1} b_2 - \dfrac{p_4 - p_3}{2} b_4 \right) = a_3 \\ b_4 \left(\dfrac{p_1 - p_4}{q_1 + 1} b_1 - \dfrac{p_2 - p_4}{q_1 - 1} b_2 + \dfrac{p_3 - p_4}{2} b_3 \right) = a_4. \end{cases} \tag{4.2}$$

Putting (4.1) into (4.2), we get

$$
\left\{
\begin{aligned}
&\frac{p_3^2+1}{a_3}\left[-\frac{p_{14}(p_1-p_4)}{(q_1+1)(q_1-1)}b_1^2+\left\{\frac{p_{24}(p_1-p_4)}{(q_1-1)^2}+\frac{p_{14}(p_2-p_4)}{(q_1+1)^2}\right\}b_1b_2\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\left.-\frac{p_{24}(p_2-p_4)}{(q_1+1)(q_1-1)}b_2^2\right]=\frac{p_{34}^2}{2} \\
&-\frac{p_4^2+1}{a_4}\left[-\frac{p_{13}(p_1-p_3)}{(q_1+1)(q_1-1)}b_1^2+\left\{\frac{p_{13}(p_2-p_3)}{(q_1-1)^2}+\frac{p_{23}(p_1-p_3)}{(q_1+1)^2}\right\}b_1b_2\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\left.-\frac{p_{23}(p_2-p_3)}{(q_1+1)(q_1-1)}b_2^2\right]=\frac{p_{34}^2}{2}.
\end{aligned}
\right. \tag{4.3}
$$

If this equation has more than 4 solutions (b_1, b_2) such that $b_1b_2 \neq 0$, then it holds that

$$
\left\{
\begin{aligned}
\frac{p_3^2+1}{a_3}\left\{-\frac{p_{14}(p_1-p_4)}{(q_1+1)(q_1-1)}\right\} &= -\frac{p_4^2+1}{a_4}\left\{-\frac{p_{13}(p_1-p_3)}{(q_1+1)(q_1-1)}\right\} \\
\frac{p_3^2+1}{a_3}\left\{-\frac{p_{24}(p_2-p_4)}{(q_1+1)(q_1-1)}\right\} &= -\frac{p_4^2+1}{a_4}\left\{-\frac{p_{23}(p_2-p_3)}{(q_1+1)(q_1-1)}\right\},
\end{aligned}
\right.
$$

namely

$$
\begin{pmatrix} p_{13}(p_1-p_3) & p_{14}(p_1-p_4) \\ p_{23}(p_2-p_3) & p_{24}(p_2-p_4) \end{pmatrix}
\begin{pmatrix} \dfrac{a_3}{p_3^2+1} \\ \dfrac{a_4}{p_4^2+1} \end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

Hence

$$
0=\det\begin{pmatrix} p_{13}(p_1-p_3) & p_{14}(p_1-p_4) \\ p_{23}(p_2-p_3) & p_{24}(p_2-p_4) \end{pmatrix} = p_{12}p_{34}(p_1-p_2)(p_3-p_4).
$$

This contradicts our assumption. Therefore, the equation (4.3) has at most 4 solutions (b_1, b_2) such that $b_1b_2 \neq 0$. Note that if $(b_1, b_2) = (\beta_1, \beta_2)$ is a solution, then $(b_1, b_2) = -(\beta_1, \beta_2)$ is also a solution and both of these solutions give the same Weierstrass data. Therefore it is clear that, for any conformal class chosen above, the number of 4-end catenoids realizing the given $(\boldsymbol{v}, \boldsymbol{a})$ is at most 2. Now we get the estimate $N_C(\boldsymbol{v}, \boldsymbol{a}) \leq N_q(\boldsymbol{v}) \times 2 \leq 4$. □

Corollary 4.3. *Any* 4*-end catenoid of TYPE II is isolated in the sense of Rosenberg* [Rose].

Proof. By the same reason as in the proof of Corollary 3.4, our assertion is clear. □

Remark 4.4. In Lemma 4.1, $N_q(\boldsymbol{v}) \leq 1$ if one of the following condition holds:

(1) $p_{jk} = 0$, i.e. $-v_j = v_k$ for some $j \neq k$;
(2) $D := {p_{12}}^2{p_{34}}^2 + {p_{13}}^2{p_{24}}^2 + {p_{14}}^2{p_{23}}^2 - 2(p_{13}p_{24}p_{14}p_{23} + p_{12}p_{34}p_{14}p_{23} + p_{12}p_{34}p_{13}p_{24}) = 0$.

Indeed, it holds that

$$\Phi_{II}(q_1+q_1^{-1}) = \begin{cases} 4(p_{13}p_{24}-p_{14}p_{23})(q_1+q_1^{-1})+8(p_{13}p_{24}+p_{14}p_{23}) & \text{if} \quad p_{12}p_{34}=0 \\ \{p_{12}p_{34}(q_1+q_1^{-1})-4p_{14}p_{23}+2p_{12}p_{34}\}\,\dfrac{(q_1-1)^2}{q_1} & \text{if} \quad p_{13}p_{24}=0 \\ \{p_{12}p_{34}(q_1+q_1^{-1})+4p_{13}p_{24}-2p_{12}p_{34}\}\,\dfrac{(q_1+1)^2}{q_1} & \text{if} \quad p_{14}p_{23}=0 \\ \{p_{12}p_{34}(q_1+q_1^{-1})+2p_{13}p_{24}-2p_{14}p_{23}\}^2\,\dfrac{1}{p_{12}p_{34}} & \text{if} \quad D=0. \end{cases}$$

Therefore, by the proof of Theorem 4.2, if either (1) or (2) holds, then we get the estimate $N_C(\boldsymbol{v},\boldsymbol{a}) \leq N_q(\boldsymbol{v}) \times 2 \leq 2$.

Moreover, we can find an additional obstruction for the existence of 4-end catenoids of TYPE II.

Theorem 4.5. *There are no 4-end catenoids $x\colon \hat{\mathbf{C}}\setminus\{q_1,q_2,q_3,q_4\} \to \mathbf{R}^3$ such that $\nu(q_j)=v_j$ and $w(q_j)=a_j$ $(j=1,2,3,4)$ if $-v_1=v_2$ and $v_3=v_4\neq \pm v_1$.*

Proof. Set $p_j := \sigma(v_j)$ as before. When our assumption holds, we may assume p_1,p_2,p_3,p_4 are nonzero real numbers satisfying $p_1p_2+1=0$ and $p_3=p_4\neq p_1,-p_1^{-1}$ without loss of generality. Suppose there exists a 4-end catenoid $x\colon \hat{\mathbf{C}}\setminus\{q_1,q_2,q_3,q_4\}\to\mathbf{R}^3$ such that $\nu(q_j)=v_j$ and $w(q_j)=a_j$ $(j=1,2,3,4)$ for some nonzero real numbers $a_1,\dots,a_n$ satisfying $a_1=a_2$ and $a_3+a_4=0$. Then it follows from direct computation that

$$0=\det A=\left\{\frac{(p_1p_3+1)(p_1-p_3)(q_1-q_2)(q_3-q_4)}{p_1(q_1-q_3)(q_2-q_3)(q_1-q_4)(q_2-q_4)}\right\}^2.$$

This contradicts our assumption $p_3\neq p_1,-p_1^{-1}$ $\square$

Next, we consider the case of TYPE I, namely, the case when all of the ends are parallel. In this case, there exist additional obstructions for an arbitrary n. For instance, since the degree of the Gauss map must be less than n for any n-end catenoid (see Section 2), the flux data

$$v_1=\cdots=v_n \tag{4.4}$$

cannot be realized by any n-end catenoid. Moreover, the following obstructions can be also found by using (2.26).

$$-v_1=-v_2=v_3=\cdots=v_n \tag{4.5}$$

$$-v_1=v_2=\cdots=v_n, \quad \textstyle\sum_{j=2}^{n-1}\sum_{k=j+1}^{n}a_ja_k\neq 0. \tag{4.6}$$

The obstruction (4.6) follows from the compatibility condition in Perez [Per]. Conversely, in the exceptional case of the obstruction (4.6), we have the following

Lemma 4.6. *Let $\boldsymbol{v}=\{v_1,\dots,v_n\}$ be a family of unit vectors in $\mathbf{R}^3$, and $\boldsymbol{a}=\{a_1,\dots,a_n\}$ a set of nonzero real numbers satisfying $-v_1=v_2=\cdots=v_n$, $a_1=\sum_{j=2}^n a_j$ and $\sum_{j=2}^{n-1}\sum_{k=j+1}^n a_ja_k=0$. Then there exists an n-end catenoid*

$x\colon \hat{\mathbf{C}}\setminus\{q_1,\dots,q_n\}\to\mathbf{R}^3$ such that $\nu(q_j)=v_j$ and $w(q_j)=a_j$ $(j=1,\dots,n)$ if and only if there are complex numbers $q_1,\dots,q_n$ satisfying

$$\sum_{\substack{k=2\\k\neq j}}^{n}\frac{a_k}{q_k-q_j}=0\qquad(j=2,\dots,n). \tag{4.7}$$

Proof. Set $p_j:=\sigma(v_j)$ as before. We may assume $p_1=q_1=\infty$ and $p_2=\dots=p_n=0$ without loss of generality. Suppose there exists an n-end catenoid $x\colon \hat{\mathbf{C}}\setminus\{q_1,\dots,q_n\}\to\mathbf{R}^3$ such that $\nu(q_j)=v_j$ and $w(q_j)=a_j$ $(j=1,\dots,n)$. Then, by (2.27) with the assumption $p_1=q_1=\infty$ in place of $p_n=q_n=\infty$, it holds that

$$b_jb_1=a_j,\quad \sum_{\substack{k=2\\k\neq j}}^{n}\frac{b_k}{q_k-q_j}=0\qquad(j=2,\dots,n).$$

Hence we have

$$0=b_1\sum_{\substack{k=2\\k\neq j}}^{n}\frac{b_k}{q_k-q_j}=\sum_{\substack{k=2\\k\neq j}}^{n}\frac{a_k}{q_k-q_j}\qquad(j=2,\dots,n).$$

Conversely, suppose there are n complex numbers $q_1,\dots,q_n$ satisfying (4.7). Put

$$f(z):=\sum_{j=2}^{n}\frac{a_j}{z-q_j},$$

and set, for any nonzero complex number t,

$$g_t(z):=-\frac{1}{tf(z)},\quad \omega_t:=-t(f(z))^2dz. \tag{4.8}$$

Then, for any t, the surface $x_t\colon \hat{\mathbf{C}}\setminus\{q_1,\dots,q_n\}\to\mathbf{R}^3$ represented by these data is an n-end catenoid such that $\nu(q_j)=v_j$, $w(q_j)=a_j$ $(j=1,\dots,n)$ and the induced metric

$$ds_t^2=\frac{(|tf|^2+1)^2}{|t|^2}|dz|^2$$

is non-degenerate. □

By the proof of Lemma 4.6, if there are n complex numbers $q_1,\dots,q_n$ satisfying (4.7), then there exists a 1-parameter family $\{x_t;\ t\in\hat{\mathbf{C}}\setminus\{0\}\}$ of n-end catenoids with the same $(\boldsymbol{v},\boldsymbol{a})$. However, when $|t|=|t'|$, x_t can be transformed to $x_{t'}$ by a certain rotation around the x_3-axis. On the other hand, in the case $n>2$, when $|t|\neq|t'|$, clearly x_t and $x_{t'}$ are not isometric to each other. Hence, the family $\{x_t;\ t>0\}$ is a non-trivial deformation.

Note that this deformation is an example of the deformation described in Lopez-Ros [LR]. It is clear that x_t is deformable in the sense of Rosenberg [Rose].

By solving the equation (4.7) with $n=4$ and 5, we have the following examples which completes the classification of at most 5-end catenoids of parallel ends. Indeed, by virtue of the conditions (4.4), (4.5) and (4.6), it is clear that

any n-end catenoid of TYPE I with $n \leq 5$ coincides with the (2-end) catenoid or one of the surfaces in Examples 4.7 and 4.8.

Example 4.7. Let $\boldsymbol{v} = \{v_1, v_2, v_3, v_4\}$ be a family of unit vectors in $\mathbf{R}^3$ satisfying $-v_1 = v_2 = v_3 = v_4$. For any set $\boldsymbol{a} = \{a_1, a_2, a_3, a_4\}$ of nonzero real numbers satisfying $a_1 = a_2 + a_3 + a_4$ and $a_2a_3 + a_2a_4 + a_3a_4 = 0$, there exist a unique 1-parameter family $\{x_t \colon \hat{\mathbf{C}} \setminus \{q_1, q_2, q_3, q_4\} \to \mathbf{R}^3; t > 0\}$ of 4-end catenoids such that $\nu(q_j) = v_j$ and $w(q_j) = a_j$ $(j = 1, 2, 3, 4)$. Indeed their representations are given by (4.8) with

$$f(z) := \frac{a_2}{z} + \frac{a_3}{z-1} + \frac{a_4}{z + \dfrac{a_4}{a_3}},$$

up to rigid motion in $\mathbf{R}^3$. Figure 4.1 shows the image of x_t for various value of t, when $a_2 = -1$ and $a_3 = a_4 = 2$.

(a) $t = 0.5$ (b) $t = 1$ (c) $t = 2$

FIGURE 4.1. Example 4.7 for $a_2 = -1$ and $a_3 = a_4 = 2$

Example 4.8. Let $\boldsymbol{v} = \{v_1, v_2, v_3, v_4, v_5\}$ be a family of unit vectors in $\mathbf{R}^3$ satisfying $-v_1 = v_2 = v_3 = v_4 = v_5$. For any set $\boldsymbol{a} = \{a_1, a_2, a_3, a_4, a_5\}$ of nonzero real numbers satisfying $a_1 = a_2 + a_3 + a_4 + a_5$ and $a_2a_3 + a_2a_4 + a_2a_5 + a_3a_4 + a_3a_5 + a_4a_5 = 0$, there exist two 1-parameter families $\{x_{t,\pm} \colon \hat{\mathbf{C}} \setminus \{q_1, q_2, q_3, q_4, q_5\} \to \mathbf{R}^3; t > 0\}$ of 5-end catenoids such that $\nu(q_j) = v_j$ and $w(q_j) = a_j$ $(j = 1, 2, 3, 4, 5)$. Indeed their representations are given by (4.8) with

$$f(z) := \frac{a_2}{z} + \frac{a_3}{z-1} + \frac{a_4}{z - \dfrac{a_2 + a_5\zeta_6}{a_2 + a_3 + a_5}} + \frac{a_5}{z - \dfrac{a_2 + a_4\bar{\zeta}_6}{a_2 + a_3 + a_4}},$$

up to rigid motion in $\mathbf{R}^3$, where ζ_6 is a primitive root of the equation $z^6 = 1$, i.e. $\zeta_6 = (1 \pm \sqrt{3}i)/2$. Remark here that $x_{t,+}$ and $x_{t,-}$ lie on the symmetric positions with respect to the x_1x_3-plane, and hence they are isometric with each other.

We also have the following

Example 4.9. Let m be an integer greater than 1, and ζ_m a primitive root of the equation $z^m = 1$. For any positive number t, define the surface $x_t \colon \hat{\mathbf{C}} \setminus \{\infty, 0, 1, \zeta_m, \dots, \zeta_m^{m-1}\} \to \mathbf{R}^3$ by the data (4.8) with

$$f(z) := \frac{(m+1)z^m + (m-1)}{z(z^m - 1)}.$$

Then $\{x_t\}$ is a 1-parameter family of $\mathbf{Z}_m$-invariant $(m+2)$-end catenoids with the same flux data. Therefore the estimate as in Theorems 3.3 and 4.2 does not hold for n-end catenoids of TYPE I for any $n \geq 4$.

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